

ATTACHMENT A

Using the Smoothed Anchoring Method to Obtain Current Price Estimates

This appendix describes the proposed method for estimating a current price associated with each license at the close of every round. If the Bureau decides to implement the comprehensive revision of the procedures described as Option 1 in “Whether To Change The Minimum Acceptable Bid Calculation,” these “current price estimates,” as they will be called, will be used in the next round when calculating part *iii* of the minimum acceptable bid formula. Specifically, for a license, this value will be the current price estimate of the license plus $z\%$. For a package, the value will be the sum of the current price estimates of the licenses that make up the package plus $z\%$ of the sum.

The current price estimates of the licenses are based on the concept that every linear optimization problem has a dual problem that provides pricing information. We begin by presenting the FCC winner determination problem as an integer program, (P1), and then discuss its linear programming representation before displaying the dual problem of interest.

$$(P1): \quad \begin{aligned} & \max \sum_{j \in B^t} b_j x_j \\ & s.t. \quad \sum_{j \in B^t} a_{ij} x_j = 1, \quad \text{for all } i \in L \quad (1) \\ & \quad \quad x_j \in \{0, 1\}, \quad \text{for all } j \in B^t \end{aligned}$$

where B^t is the set of considered bids in round t ,

b_j is the bid amount of bid j ,

L is the set of licenses being auctioned,

$$a_{ij} = \begin{cases} 1, & \text{if license } i \text{ is in bid } j \\ 0, & \text{otherwise} \end{cases} \text{ and,}$$

$$x_j = \begin{cases} 1, & \text{if bid } j \text{ is in the winning set} \\ 0, & \text{otherwise} \end{cases}$$

In this formulation, x_j is an indicator variable that equals one if bid j is in the provisionally winning set and zero otherwise. Thus, the sum of the bid amounts of all provisionally winning bids produces the maximum obtainable revenue for round t . Constraints (1) ensure that each license is awarded exactly once. The constraints that ensure that a bidder’s bids between rounds are mutually exclusive are not represented in (P1) since they will be ignored in the linear representation of the problem.¹

¹ These constraints will be ignored in the linear program representation since they are rarely binding in the relaxation of the integer-programming problem and because adding such constraints to the dual problem creates “degeneracy” in the solution thereby causing multiple alternative solutions.

The linear program of (P1) relaxes the restriction on the variables x_j , for all $j \in B^t$, allowing these variables to take on any value between zero and one. The linear programming representation of (P1) is shown in (P2):

$$(P2): \quad \begin{aligned} & \max \sum_{j \in B^t} b_j x_j \\ & \text{s.t.} \quad \sum_{j \in B^t} a_{ij} x_j = 1, \quad \text{for all } i \in L \\ & \quad \quad x_j \geq 0, \quad \text{for all } j \in B^t \end{aligned}$$

The dual formulation of (P2) can be used to identify a price, π_i , for each license i , and is shown in the following linear program (P3):

$$(P3): \quad \begin{aligned} & \min \sum_{i \in L} \pi_i \\ & \text{s.t.} \quad \sum_{i \in L} a_{ji} \pi_i \geq b_j, \quad \text{for all } j \in B^t \setminus F \\ & \quad \quad \pi_i \geq b_j, \quad \text{for all } j \in F \end{aligned} \quad (2)$$

and i is the license index associated with bid j (3)

where $F \subset B^t$ is the set of FCC bids on each license² and,

$$a_{ji} = \begin{cases} 1, & \text{if bid } j \text{ contains license } i \\ 0, & \text{otherwise} \end{cases}.$$

The optimal value of each variable, π_i , in (P3) corresponds to a dual price³ – often called a “shadow price” – for each constraint, i.e., each license, in (P2). The dual price of each license measures the monetary cost of not awarding the license to whom it has been provisionally assigned under the solution to (P2). Thus, this monetary cost has a clear and natural use in estimating the current price of a license given the bids considered in the current round.

Constraints (2) in (P3) ensure that the dual price of a license must be at least as large as the greatest bid made on that license. For a package, these constraints ensure that the sum of the dual prices of the licenses that make up a particular package must be at least as large as the greatest bid made on that package. Constraints (3) in (P3) ensure that

² The bid amount for a FCC bid is some small amount less than the minimum-opening bid for that license. See Auction Of Licenses in the 747-762 and 777-792 MHz Bands Scheduled for March 6, 2001; Modifications to the Calculation for Determining Minimum Acceptable Bids and the Provisions Concerning “Last and Best Bids” and Other Procedural Issues, DA 01-12, *Public Notice*, 16 FCC Rcd 217 (2001)(Section III discusses the reasons for this approach).

³ We note that for non-linear problems, these dual prices are also known as *Lagrange multipliers*.

if a license has not been bid on, the dual price of that license is at least as large as the FCC bid amount.

Ideally, the solution to (P2) is identical to the solution of (P1). When this occurs, the sum of the dual prices of the licenses comprising any provisionally winning bid equals the winning bid amount. However, (P2) is only an approximation to the integer problem⁴ and often *overestimates* the maximum revenue of (P1). When this occurs, the sum of the dual prices of the licenses in at least one provisionally winning bid will be greater than the respective bid amount. Thus, using the dual prices of (P3) can result in minimum acceptable bid amounts that are too high.

We propose to resolve this issue by using *pseudo-dual prices*,⁵ rather than the dual prices of (P3). These pseudo-dual prices are obtained by forcing the sum of the dual prices of the licenses comprising a provisionally winning bid to equal its respective bid amount. For example, suppose there are two bids in the provisionally winning set in round t : a bid on license A for \$10 and a bid on package BC for \$25. The pseudo-dual price of A would exactly equal \$10 and the sum of the pseudo-dual prices of B and C would exactly equal \$25. These restrictions ensure that the sum of the pseudo-dual prices equals the maximum revenue for the round (e.g. \$35) and that minimum acceptable bid amounts reflect the bid amounts of bids in the provisionally winning set.

Pseudo-dual prices for each license i , denoted π_i , satisfy the following constraints:

$$\sum_{i \in L} a_{ji} \pi_i + \delta_j \geq b_j, \text{ for all } j \in B^t \setminus (W^t \cup F) \quad (4)$$

$$\sum_{i \in L} a_{ji} \pi_i = b_j, \quad \text{for all } j \in W^t \quad (5)$$

$$\pi_i \geq b_j, \quad \text{for all } j \in F \setminus (W^t \cap F) \quad (6)$$

and i is the license index associated with bid j

$$\delta_j \geq 0, \quad \text{for all } j \in B^t \setminus (W^t \cup F) \quad (7)$$

where $W^t \subset B^t$ is the provisionally winning bid set in round t and,

δ_j is a slack variable that represents the difference between the bid amount of non-winning bid j and the sum of pseudo-dual prices of the licenses contained in non-winning bid j

⁴ When the problem is a convex optimization problem, the primal and dual problems yield the same objective function values. This is called strong-duality. These conditions do not hold for integer programming problems, often resulting in a gap between the linear programming and integer programming solution values.

⁵ In our research we found this term first applied to auction pricing in the paper by Rassenti, Smith and Bulfin (1982), "A combinatorial auction mechanism for airport slot allocation," *Bell Journal of Economics*, vol. 13, pp. 402-417.

Constraints (5) ensure that for each provisionally winning bid, the sum of the dual prices of the licenses comprising that bid equal its respective bid amount. This new restriction requires that we ease restriction (2) in (P3) for non-winning bids in order to ensure that a feasible solution exists. Constraints (4) provide this needed slack. Constraints (6) are equivalent to constraints (3) in (P3) and constraints (7) force the slack variables to be non-negative.

Satisfying constraints (5) implies that the sum of the pseudo-dual prices always yields the maximum revenue for the round. There are likely to be many sets of pseudo-dual prices that satisfy this constraint set. For instance, in the example provided earlier, the pseudo-dual prices of B and C might be any two numbers that together sum to \$25.

By keeping constraints (4)-(7), we have the flexibility to choose an objective function that will help in selecting among multiple solutions while still ensuring that the sum of the pseudo-dual prices yields the maximum revenue of the round. We would like an objective function that minimizes the values of the slack variables δ_j , for all $j \in B^t \setminus (W^t \cup F)$ in order to obtain pseudo-dual prices that are close to the dual prices of (P3). We have tested a number of alternative objective functions:

1. Minimization of the maximum δ_j for all $j \in B^t \setminus (W^t \cup F)$ followed by maximization of the minimum π_i for all i in license set L , in an iterative manner. (DeMartini, Kwasnica, Ledyard and Porter, 1999)
2. Minimization of the sum of the squares of δ_j for all $j \in B^t \setminus (W^t \cup F)$. (also DeMartini, Kwasnica, Ledyard and Porter, 1999)
3. Minimization of the sum of the δ_j for all $j \in B^t \setminus (W^t \cup F)$ using a “centering” algorithm⁶ to solve, essentially finding an average among all sets of optimal pseudo-dual prices.

In testing the above alternatives, we frequently observed instances where the pseudo-dual price of a license significantly changed from round to round. We acknowledge that prices of licenses should be allowed to reflect real changes, both increases and decreases, in the way bidders value the licenses over time. However, we believe that large oscillations in minimum acceptable bid amounts for the same bid that are due to irrelevant factors such as multiple optimal solutions, can be confusing to bidders. We have therefore chosen a method that attempts to balance minimizing the slack variables and reducing the fluctuations in pseudo-dual prices from round to round. This method requires solving two optimization problems, the first of which is alternative 3 above, which we present as (P4):

⁶ The centering algorithm used in this testing was the barrier method available in CPLEX, a commercial optimization package.

$$\begin{aligned}
\Omega^* &= \min \sum_{j \in B^t \setminus (W^t \cup F)} \delta_j \\
s.t. \quad & \sum_{i \in L} a_{ji} \pi_i + \delta_j \geq b_j, \text{ for all } j \in B^t \setminus (W^t \cup F) \\
(P4): \quad & \sum_{i \in L} a_{ji} \pi_i = b_j, \quad \text{for all } j \in W^t \\
& \pi_i \geq b_j, \quad \text{for all } j \in F \setminus (W^t \cap F) \\
& \quad \quad \quad \text{and } i \text{ is the license index associated with bid } j \\
& \delta_j \geq 0, \quad \text{for all } j \in B^t \setminus (W^t \cup F)
\end{aligned}$$

Since multiple optimal solutions can exist to (P4) we solve a second optimization problem that chooses a solution in a way that reduces the magnitude of price fluctuations between rounds. Specifically, we use an objective function that applies the concepts of exponential smoothing⁷ to choose among alternative pseudo-dual prices with the additional constraint on the problem that the sum of the slack variables equals Ω^* (the optimal value of (P4)). This objective function minimizes the sum of the squared distances of the resulting pseudo-dual prices in round t from their respective smoothed prices in round $t-1$.⁸ At the start of the auction, we use the minimum opening bid prices as the prior smoothed prices. Since these opening prices are based on bandwidth and population, the pricing algorithm begins with *a priori* information about the differences among licenses.

Let π_i^t be the pseudo-dual price of license i in round t . The smoothed price for license i in round t is calculated using the following exponential smoothing formula:

$$p_i^t = \alpha \pi_i^t + (1 - \alpha) p_i^{t-1}$$

where p_i^{t-1} is the smoothed price in round $t-1$,

$$0 \leq \alpha \leq 1, \text{ and}$$

$$p_i^0 = \text{the minimum opening bid amount for license } i.$$

Consistent with prior practice of the Commission, a weighting factor of $\alpha = 0.5$ has been chosen but can change, as the Commission requires.

⁷ Exponential smoothing often is used in determining minimum acceptable bids in FCC auctions. See, e.g., Auction of Licenses in the 747-762 and 777-792 MHz Bands; Auction Notice and Filing Requirements for 12 Licenses in the 700 MHz Bands Auction Scheduled for May 10, 2000; Minimum Opening Bids and Other Procedural Issues, DA 00-292, *Public Notice*, 15 FCC Rcd 2921, Attachment G (2000).

⁸ This objective function is a convex, quadratic function. This quadratic optimization problem is solved using the barrier method.

The following quadratic program (QP) will find the pseudo-dual price, π_i^t , for each license i in round t that minimizes the sum of the squared distances from the respective smoothed price in round $t-1$ while ensuring that the pseudo-dual prices sum up to the provisionally winning bid amounts and that the sum of the slack variables is minimized.

$$\begin{aligned}
 \text{(QP):} \quad & \min \sum_{i \in L} (\pi_i^t - p_i^{t-1})^2 \\
 & s.t. \quad \sum_{i \in L} a_{ji} \pi_i^t + \delta_j \geq b_j, \quad \text{for all } j \in B^t \setminus (W^t \cup F) \\
 & \quad \sum_{i \in L} a_{ji} \pi_i^t = b_j, \quad \text{for all } j \in W^t \\
 & \quad \sum_{j \in B^t \setminus (W^t \cup F)} \delta_j = \Omega^* \\
 & \quad \pi_i^t \geq b_j, \quad \text{for all } j \in F \setminus (W^t \cap F) \\
 & \quad \quad \quad \text{and } i \text{ is the license index associated with bid } j \\
 & \quad \delta_j \geq 0, \quad \text{for all } j \in B^t \setminus (W^t \cup F)
 \end{aligned}$$

The objective function of (QP) minimizes the difference between the current round's pseudo-dual price and the previous round's smoothed price, where p_i^{t-1} is known and treated as a constant within the optimization.⁹

Among alternative prices that satisfy all constraints, the objective function of this optimization problem chooses one that forces the pseudo-dual prices to be as close as possible to the previous round's smoothed price. Thus, we call this the *Smoothed Anchoring Method* since we “anchor” on the smoothed prices when solving for the pseudo-dual prices. We define the “current price estimate” for license i in round t as the pseudo-dual price, π_i^t , obtained by solving (QP).

If the Bureau decides to implement the comprehensive revision of the procedures described as Option 1 in “Whether To Change The Minimum Acceptable Bid Calculation,” the minimum acceptable bid amount for a license in round $t+1$ under part *iii* will be the current price estimate of the license plus $z\%$. For a package, the value will be the sum of the current price estimates of the licenses that make up the package plus $z\%$ of the sum.

⁹ Once the pseudo-dual prices, π_i^t , have been determined, the smoothed prices, p_i^t , can be calculated and used for solving (QP) in round $t+1$.